A Note on Intuitionistic Fuzzy Logics

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1. Introduction

The purpose of this note it to outline the salient features of intuitionistic fuzzy logic, which is a branch of symbolic logic, and to dispel any misgivings that it is somehow incompatible with traditional Aristotelian logic.

Leibniz is often considered the remote founder of mathematical logic, although it was not until the nineteenth century that widely accepted attempts were made to express formal logic in the manner of algebraic theorems. George Boole is usually regarded as the father of this symbolic logic (Sanguineti, 1992: 16-18). His work was extended by Frege «who arrived at the predicate calculus which turned out to be an adequate logical basis for all of today's mathematics» (Crossley *et al*, 1972:1).

Symbolic logic was developed independently of mathematics early in the twentieth century. It became a specialized field in which detailed axiomatic systems were formulated. Some of these came to be related to the foundations of mathematics. More particularly, Hilbert, Gödel and Tarski studied the value and limits of axiomatization, the relation between logic and mathematics, and the problem of truth. It would take us from our purpose to digress into this, but the drift to the increasingly common view of truth as relative has been well documented by Westbrook (1991): truth and validity are often confused and mathematics is sometimes seen as only an axiomatic system.

The key notion is that of a set, which in naive set theory is defined as a collection of objects. Axiomatic set theory is a more sophisticated endeavour which was devised to avoid the antinomies observed by Russell and others (Russell, 1937: 115, 158). For our present purposes we do not need to delve into these refinements.

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2. Fuzzy Logic

More recently, the notion of set membership has been extended to include varying degrees of membership. Whereas in the set theory of traditional mathematical logic an object either belongs to or does not belong to any particular set, in the fuzzy logic of Zadeh (1965) the membership function can vary in value between 0 and 1 (inclusively).

Bertrand Russell (1923) foreshadowed the development of fuzzy logic in a one-off paper, but did not return to the topic. However, he did sow the seeds of misunderstanding which has bedevilled relations between classical philosophers and symbolic logicians when he asserted that «all traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life, but only to an imagined celestial one. The law of the excluded middle [A or not-A] is true when precise symbols are employed but it is not true when symbols are vague, as, in fact, all symbols are». Symbols are not vague: their incorrect use may be vague, and this includes attributing more to them than they can represent.

Each proposition (in the classical sense) can be assigned a truth value: truth - 1, falsity - 0. Thus one might define "warmth", for example, as a temperature between, but not including 18ºC and 25ºC. (Heat and temperature are not the same, of course, but we are merely trying to utilise an apt illustration (Shannon, 1968).) With this (arbitrary) definition, 20ºC would be warm, but 15ºC or even 25ºC would not be warm.

In fuzzy logics this truth value is a real number in the interval [0,1] and called the "degree of truth" of a particular proposition. With our temperature example, one might say that though 19ºC and 20ºC are each warm, 20ºC is clearly warmer (in some sense) than 19ºC. Thus there are degrees of warmth.

There is nothing in this to contradict either traditional two-valued logic or the Law of Non-contradiction [not-(A and not-A)] (which is logically equivalent to the Law of the Excluded Middle through de Morgan's Laws). As well as Russell, some current proponents of fuzzy logic, such as Kosko (1993:30-33), claim otherwise. They cite, for example, the case where the degree of membership and the degree of non-membership of a set each equals 0.5 as a case of A equal to not-A. Intuitionistic fuzzy logic, described in the next section, can avoid this because there the degree of non-membership is not necessarily equal to one minus the degree of membership. More fundamentally though, traditional set theory, in so far as it represents Aristotelian logic, is a mapping to the set of integers 0 and 1, whereas fuzzy set theory is a mapping onto the closed subset [0,1] of the real numbers.

Nor is there any necessary relation with many-valued logics. Membership and degrees of membership are themselves two distinct ideas. In the example cited here the degrees of membership are a mathematical concept related to Dedekind cuts of the real number continuum which in turn is related to the experimental ability to distinguish the temperatures of physical objects. Thus fuzzy logic can be used as a heuristic for logical inference.

3. Intuitionistic Fuzzy Logics

A more recent extension of these ideas may be found in Intuitionistic Fuzzy Logic (IFL) (Atanssov, 1988), which is a modification of Intuitionistic Fuzzy Sets (IFSs) (Atanassov, 1983, 1986). One more value is added: the "degree of falsity", which is also in the interval [0,1]. Thus one assigns to the proposition p two real numbers $\mu(p)$ and $\gamma(p)$ with the constraint: $0 \leq \mu(p) + \gamma(p) \leq 1$.

This assignment applies to an evaluation function V:

$$V(p) = <\mu(p), \gamma(p)>.$$

The evaluation of the negation is then

$$V(\neg p) = <\gamma(p), \mu(p)>.$$

Thus in a sense with our temperature example, $10^{\circ}C$ is less of a member, or more of a non-member, of the set of warm things than $17^{\circ}C$. This gives us more flexibility in simulation in mathematical modelling through generalized nets (Shannon *et al*, 1996), or, to put it another way, IFL can model the situation where two people who disagree with a third do not necessarily agree with each other.

IFL contains all the logical operators associated with classsical symbolic logic. They are outlined here together with seminal references for any interested reader who wishes to pursue them further. When the values $V(p)$ and $V(q)$ of the propositions p and q are known, the evaluation function V can be extended for the operations $\wedge$ (and), $\vee$ (or) and $\supset$ (if-then) by the definitions:

$$V(p \wedge q) = <\min\{\mu(p), \mu(q)\}, \max\{\gamma(p), \gamma(q)\}>,$$
$$V(p \vee q) = <\max\{\mu(p), \mu(q)\}, \min\{\gamma(p), \gamma(q)\}>,$$
$$V(p \supset q) = <\max\{\gamma(p), \mu(q)\}, \min\{\mu(p), \gamma(q)\}>.$$

Properties of the IF propositional calculus may be found in Atanassov (1998) and those of the IF predicate calculus in Atanassov (1990) and Gargov and Atanassov (1992). There the IF interpretations of universal and existential quantifiers are introduced respectively as:

$$V(\circledR A) = <\min_{a \in A} \mu(A), \max_{a \in A} \gamma(A)>,$$
$$V(\exists A) = <\max_{a \in A} \mu(A), \min_{a \in A} \gamma(A)>.$$

Intuitionistic Fuzzy Modal Logic (IFML) is introduced in Atanassov (1989). There too the greater scope of IFL is seen in the operators "necessity" (o) and "possibility" ($\Diamond$) which are respectively defined by:

$$V(op) = <\mu(p), 1 - \mu(p)>,$$
$$V(\Diamond p) = <1 - \gamma(p), \gamma(p)>.$$

These operators can be extended (Atanassov and Gargov 1990; Atanassov 1994a,b), and IF temporal logic has been developed (Atanassov, 1990). Recently, IF models have been constructed for the standard propositional and predicate calculus axiomatic systems, for the Kolmogorov, the Lukasievicz-Tarski, the Meredith and other axiomatic systems, as well as the S1-S6 modal logic systems.

Some applications of the IFL-elements may be found in IF PROLOG (Atanassov and Georgiev, 1993), IF expert systems (Atanassov, 1993a), IF constraint logic programming (Atanassov, 1993b), IF neural networks (Hadjyisky and Atanassov, 1993), and IF graphs (Shannon and Atanassov, 1994, 1995).

4. Concluding Comments

Symbolic logic can act as a bridge in certain, albeit limited, circumstances between mathematics and philosophy. The symbols can never replace thought, nor entirely represent it, as the more extreme proponents of artificial intelligence sometimes claim. However, symbolism can be "a tool of thought" (Iverson, 1980), a sentiment echoed by Alfred North Whitehead when he justified symbolism on the grounds that «by relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems» (quoted in Cajori, 1929).

Finally, in the words of Copleston (1946: 286): «Modern symbolic logic may be an addition, and a very valuable addition, to the logic of Aristotle, but it should not be regarded as a completely opposite counter thereto: it differs from non-symbolic logic by its higher degree of formalisation».

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